Hats: all or nothing

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Abstract.
N players are randomly fitted with a colored hat (q different colors). All players guess simultaneously the color of their own hat observing only the hat colors of the other $N-1$ players. The team wins if *all* players guess right. No communication of any sort is allowed, except for an initial strategy session before the game begins. In the first part of our investigation we have 2,3 or 4 different colors with equal probabilities. In the second part we have two colors where the probabilities may differ. We construct optimal strategies and maximal probability of winning the game for any number of players.

Introduction.
Hat puzzles were formulated at least since Martin Gardner's 1961 article [8]. They have got an impulse by Todd Ebert in his Ph.D. thesis in 1998 [6]. Ebert's hat problem: All players guess simultaneously the color (white or black) of their own hat observing only the hat colors of the other $N-1$ players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his hat color correctly and none of the players has an incorrect guess. Ebert's hat problem with $N=2^k-1$ players is solved in [7], using Hamming codes, and with $N=2^k$ players in [5] using extended Hamming codes. Lenstra and Seroussi [15] show that in Ebert's Hat Game, playing strategies are equivalent to binary covering codes of radius one. Ebert's asymmetric version (where the probabilities of getting a white or black hat may be different) is studied in [18],[19],[20].
In this paper N distinguishable players are randomly fitted with a colored hat (q different colors). All players guess simultaneously the color of their own hat observing only the hat colors of the other $N-1$ players. The team wins if *all* players guess his or her hat color correctly. An initial strategy session is allowed. Our goal is to maximize the probability of winning the game and to describe optimal strategies.
The results of our investigation are rather intriguing. For example 10 players, 3 colors with equal probability: guessing at random gives winning probability $3^{-10}=1/59049$, where our strategy will give probability $1/3$. Another example: 10 players, 4 colors with equal probability: guessing at random gives winning probability $4^{-10}=1/1048576$, where our strategy will give probability $1/4$.
In case of two colors with probabilities p and q the maximal winning probability is: $\frac{1+|q-p|^N}{2}$.

PART I

In this part N distinguishable players are randomly fitted with a colored hat (q different colors with equal probability). We obtain optimal strategies and probabilities when $q=2$, $q=3$ and $q=4$.

I.1 GOOD and BAD CASES.
The N persons in our game are distinguishable, so we can label them from 1 to N.
Although we are interested in the symmetric case (every color has the same probability), we will work for a great part with a generalized model: P(color k)= p_k ($k=1,..,q$; $\sum_{k=1}^{q} p_k=1$).
Each possible configuration of the hats can be represented by an element of $B=\{b_1b_2\dots b_N | b_i \in \{1,2,..,q\}, i=1,2..,N\}$.
Player i sees code $b_1..b_{i-1}b_{i+1}..b_N$ with decimal value $s_i=\sum_{k=1}^{i-1} b_k.q^{N-k-1}+\sum_{k=i+1}^{N} b_k.q^{N-k}$.
Each player has to make a choice out of q possibilities: 1='guess color 1', 2='guess color 2',…, q ='guess color q'. We define a decision matrix $D=(a_{i,j})$ where $i \in \{1,2,..,N\}$ (players); $j=s_i$; $a_{i,j} \epsilon \{1,2,\dots,q\}$ (guess color 1, color 2,…, color q).
The meaning of $a_{i,j}$ is: player i sees j and takes decision $a_{i,j}$ (guess 1 or 2 or … q).
We observe the total probability (sum) of our guesses.

For each $b_1 b_2 \dots b_N$ in B with n_k k's $(k = 1,..,q;\ \sum_{k=1}^{q} n_k = N)$ we have (start: sum=0):
CASE $b_1 b_2 \dots b_N$
IF $a_{1,s_1} = b_1$ AND $a_{2,s_2} = b_2$ AND ... AND $a_{N,s_N} = b_N$ THEN sum=sum+$p_1^{n_1} p_2^{n_2} \dots p_q^{n_q}$;
(all players guess right).
We define the Hamming distance between $p_1^{n_1} p_2^{n_2} \dots p_q^{n_q}$ and $p_1^{m_1} p_2^{m_2} \dots p_q^{m_q}$ as $\sum_{i=1}^{q} |n_i - m_i|$, where $\sum_{i=1}^{q} n_i = \sum_{i=1}^{q} m_i = N$.
Any choice of the $a_{i,j}$ in the decision matrix determines which CASES have a positive contribution to sum (a GOOD CASE) and which CASES don't contribute positive to sum (a BAD CASE).
We focus on player i $(i \in \{1,..,N\})$. Each $a_{i,j} = m$ $(m \in \{1,2,\dots,q\})$ has $(q-1)$ counterparts $a_{i,j} = k$ $(k \in \{1,2,\dots,q\} \backslash \{m\})$: use the flipping procedure in position i: CASE $b_1..b_{i-1} m b_{i+1}..b_N \rightarrow$ CASE $b_1..b_{i-1} k b_{i+1}..b_N$. So, when the GOOD CASE has probability $p_1^{n_1} p_2^{n_2} \dots p_q^{n_q}$ we get, for fixed m and k, a BAD CASE with probability $p_1^{n_1} p_2^{n_2} \dots p_q^{n_q} p_m^{-1} p_k$ and Hamming distance 2 to our GOOD CASE.

I.2 Hamming Complete Set

We start with a notation: $\sum_{n_2,..,n_q}^{*} a(n_2,..,n_q) p_2^{n_2} \dots p_q^{n_q} = \sum_{n_2,..,n_q} p_2^{n_2} \dots p_q^{n_q}$ where $a(n_2,..,n_q) \neq 0$
$\mathcal{H} = \{S_1, S_2,.., S_q\}$ where $S_i \subset \sum^{*} (p_2 + p_3 +..+ p_q)^N = \sum_{\sum_{m=2}^{q} n_m \le N} (p_2^{n_2} \dots p_q^{n_q})$ $(i = 1,2,..,q)$.
$\mathcal{H}$ is Hamming Complete when we have the following three properties:

- (i) S_i consists only of elements with Hamming distance greater than 2 to each other
- (ii) The sets S_i are disjunct
- (iii) Completeness: $\bigcup_i S_i = \sum^{*} (p_2 + p_3 +..+ p_q)^N$

If we can construct an Hamming Complete Set, then each S_i induces a strategy $\mathcal{S}_i$ with probability $\mathcal{P}_i$ (this will become clear in the next sections).
We now return to the symmetric case: each color has probability $1/q$. This is an upper bound of $\mathcal{P}_i$ (we can't do better than the result of one player), so we have: $\mathcal{P}_i \le 1/q$ $(i = 1,2,..,q)$ and $\sum_{i=1}^{q} \mathcal{P}_i = 1$.
Conclusion: All strategies $\boldsymbol{\mathcal{S}_i}$ $(\boldsymbol{i = 1,2,..,q})$ are optimal and the probability of each strategy is $\boldsymbol{1/q}$, independent of the number of players.
In the next sections we construct Hamming Complete Sets up to 4 colors.

I.3 Two color Hat Game

Hamming Complete Set:
$S_1 = \sum_{k\ even} {p_2}^k$
$S_2 = \sum_{k\ odd} {p_2}^k$
$\boldsymbol{\mathcal{S}}_1$: guess in such a way that there is an even number of hats of color 2.
$\boldsymbol{\mathcal{S}}_2$: guess in such a way that there is an odd number of hats of color 2.
$\mathcal{P}_1 = \mathcal{P}_2 = 1/2$, which agrees with
$\mathcal{P}_1 = \sum_{k\ even} \binom{N}{k} {p_1}^{N-k} {p_2}^k$ and $\mathcal{P}_2 = \sum_{k\ odd} \binom{N}{k} {p_1}^{N-k} {p_2}^k$ $(p_1 = p_2 = 1/2)$.

I.4 Three color Hat Game

Operator T is defined by $T(p_2^m p_3^n) = \sum_{s=-[\frac{n}{3}]}^{[\frac{m}{3}]} p_2^{m-3s} p_3^{n+3s}$.

$[p_1 + (p_2 + p_3)]^N = \sum_{k=0}^{N} \binom{N}{k} {p_1}^{N-k} (p_2 + p_3)^k = 1$.
Concentrating on $(p_2 + p_3)^k$, we construct an Hamming Complete Set:

$S_1 = \sum_{n=0}^{[\frac{N}{2}]} T(p_2^n p_3^n) + \sum_{n=0}^{[\frac{N-3}{2}]} T(p_2^{n+3} p_3^n)$

$S_2 = \sum_{n=0}^{\left[\frac{N-1}{2}\right]} T(p_2^{n+1}p_3^n) + \sum_{n=0}^{\left[\frac{N-2}{2}\right]} T(p_2^n p_3^{n+2})$

$S_3 = \sum_{n=0}^{\left[\frac{N-1}{2}\right]} T(p_2^n p_3^{n+1}) + \sum_{n=0}^{\left[\frac{N-2}{2}\right]} T(p_2^{n+2} p_3^n)$

(i), (ii) and (iii) are easily verified.

Conclusion: All strategies $\mathcal{S}_i$ $(i = 1, 2, .., 3)$ are optimal and the probability of each strategy is $1/3$, independent of the number of players.

We notice that $\mathcal{S}_3$ can be found by interchanging in $\mathcal{S}_2$ the colors 2 and 3.

We illustrate our theory with three examples.

Example 1: Three players and three colors.
$S_1 = \sum_{n=0}^{\left[\frac{N}{2}\right]} T(p_2^n p_3^n) + \sum_{n=0}^{\left[\frac{N-3}{2}\right]} T(p_2^{n+3} p_3^n) = T(1) + T(p_2p_3) + T(p_2^3) = 1 + p_2p_3 + (p_2^3 + p_3^3)$
with probability $p_1^3 + 6p_1p_2p_3 + (p_2^3 + p_3^3)$, inducing strategy $\mathcal{S}_1 = \{300, 111, 030, 003\}$: when you see two identic colors, guess that color, otherwise choose the missing color.
$S_2 = \sum_{n=0}^{\left[\frac{N-1}{2}\right]} T(p_2^{n+1}p_3^n) + \sum_{n=0}^{\left[\frac{N-2}{2}\right]} T(p_2^n p_3^{n+2}) = T(p_2) + T(p_2^2p_3) + T(p_3^2) = p_2 + p_2^2p_3 + p_3^2$, with probability $3p_1^2p_2 + 3p_2^2p_3 + 3p_1p_3^2$, inducing strategy $\mathcal{S}_2$ ={210, 021, 102}. $\mathcal{S}_3 = \{201, 012, 120\}$.

Example 2: Four players and three colors.
$S_1 = \sum_{n=0}^{\left[\frac{N}{2}\right]} T(p_2^n p_3^n) + \sum_{n=0}^{\left[\frac{N-3}{2}\right]} T(p_2^{n+3} p_3^n) = T(1) + T(p_2p_3) + T(p_2^2p_3^2) + T(p_2^3) = 1 + p_2p_3 + p_2^2p_3^2 + (p_2^3 + p_3^3)$, probability $p_1^4 + 12p_2p_3 + 6p_2^2p_3^2 + 4p_1(p_2^3 + p_3^3)$ which gives strategy $\mathcal{S}_1$ ={400, 211, 022, (130, 103)}.
$S_2 = \sum_{n=0}^{\left[\frac{N-1}{2}\right]} T(p_2^{n+1}p_3^n) + \sum_{n=0}^{\left[\frac{N-2}{2}\right]} T(p_2^n p_3^{n+2}) = T(p_2) + T(p_2^2p_3) + T(p_3^2) + T(p_2p_3^3) = p_2 + p_2^2p_3 + p_3^2 + (p_2p_3^3 + p_2^4)$, probability $4p_1^3p_2 + 12p_1p_2^2p_3 + 6p_1^2p_3^2 + 4p_2p_3^3 + p_2^4$;
$\mathcal{S}_2 = \{320, 121, 202, 013, 040\}$; $\mathcal{S}_3 = \{302, 112, 220, 031, 004\}$

Example 3: Ten players and three colors.
We limit to:
$S_1 = \sum_{n=0}^{\left[\frac{N}{2}\right]} T(p_2^n p_3^n) + \sum_{n=0}^{\left[\frac{N-3}{2}\right]} T(p_2^{n+3} p_3^n) = T(1) + T(p_2p_3) + T(p_2^2p_3^2) + T(p_2^3p_3^3) + T(p_2^4p_3^4) + T(p_2^5p_3^5) + T(p_2^3) + T(p_2^4p_3) + T(p_2^5p_3^2) + T(p_2^6p_3^3) =$
$1 + p_2p_3 + p_2^2p_3^2 + (p_2^6 + p_2^3p_3^3 + p_3^6) + (p_2^7p_3 + p_2^4p_3^4 + p_2p_3^7) + (p_2^8p_3^2 + p_2^5p_3^5 + p_2^2p_3^8) +$
$(p_2^3 + p_3^3) + (p_2^4p_3 + p_2p_3^4)$+($p_2^5p_3^2 + p_2^2p_3^5$)+($p_2^9 +\ p_2^6p_3^3 + p_2^3p_3^6 + p_3^9$)
Strategy $\mathcal{S}_1$ ={10.0.0, 811, 622, (460, 433, 406), (271, 244, 217), (082, 055, 028), (730, 703), (541, 514), (352, 325), (190, 163, 136, 109)}.

There is a way to determine the optimal strategies without using the operator T.
We give the key for strategy $\mathcal{S}_1$:
$\sum_{k\ even} \binom{N}{k} \sum_{s=0}^{\left[\frac{k}{3}\right]} \binom{k}{3s + k_2} + \sum_{k\ odd} \binom{N}{k} \sum_{s=0}^{\left[\frac{k}{3}\right]} \binom{k}{3s + k_1} = 3^{N-1}$
where $k_1 = \frac{(k-3)mod6}{2}$ and $k_2 = \frac{kmod6}{2}$

Taking N =3, we get: $\binom{3}{0}\binom{0}{0} + \binom{3}{2}\binom{2}{1} + \binom{3}{3}\{\binom{3}{0} + \binom{3}{3}\}$, inducing strategy {300, 111, 030, 003}.

When N =4 we get: $\binom{4}{0}\binom{0}{0}+\binom{4}{2}\binom{2}{1}+\binom{4}{4}\binom{4}{2}+\binom{4}{3}\{\binom{3}{0}+\binom{3}{3}\}$, inducing strategy {400, 211, 022, 130, 103}.

When $N = 10$ we get:
$\binom{10}{0}\binom{0}{0}+\binom{10}{2}\binom{2}{1}+\binom{10}{4}\binom{4}{2}+\binom{10}{6}\left\{\binom{6}{0}+\binom{6}{3}+\binom{6}{6}\right\}+..+\binom{10}{9}\left\{\binom{9}{0}+\binom{9}{3}+\binom{9}{6}+\binom{9}{9}\right\}$.
$\mathcal{S}_1 =$ {10.0.0, 811, 622, (460, 433, 406), …, (190, 163, 136, 109)}.

I.5 Four color Hat Game

$[p_1 + (p_2 + p_3 + p_4)]^N = \sum_{k=0}^{N} \binom{N}{k} {p_1}^{N-k} (p_2 + p_3 + p_4)^k = 1$

Concentrating on $(p_2 + p_3 + p_4)^k$, we construct an Hamming Complete Set:

$S_1 = \sum_{n=0}^{[\frac{N}{2}]}(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-3}{2}]} p_2p_3p_4(p_2^2 + p_3^2 + p_4^2)^n$

$S_2 = \sum_{n=0}^{[\frac{N-1}{2}]} p_2(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-2}{2}]} p_3p_4(p_2^2 + p_3^2 + p_4^2)^n$

$S_3 = \sum_{n=0}^{\left[\frac{N-1}{2}\right]} p_3(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{\left[\frac{N-2}{2}\right]} p_2p_4(p_2^2 + p_3^2 + p_4^2)^n$

$S_4 = \sum_{n=0}^{[\frac{N-1}{2}]} p_4(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-2}{2}]} p_2p_3(p_2^2 + p_3^2 + p_4^2)^n$

(i), (ii) are easily verified.

To prove completeness, we first consider the sum of coefficients of $(p_2 + p_3 + p_4)^{2k}$:

$\sum_{i+j+l=k}\binom{2k}{2i, 2j, 2l} + 3\sum_{i+j+l=k-1}\binom{2k}{2i, 2j+1, 2l+1} = (1+1+1)^{2k} = 3^{2k}$

The sum of coefficients of $(p_2 + p_3 + p_4)^{2k+1}$:

$\sum_{i+j+l=k-1}\binom{2k+1}{2i+1, 2j+1, 2l+1} + 3\sum_{i+j+l=k}\binom{2k+1}{2i, 2j, 2l+1} = 3^{2k+1}$

Conclusion: All strategies $\mathcal{S}_i$ $(i = 1, 2, .., 4)$ are optimal and the probability of each strategy is $1/4$, independent of the number of players.

We notice that $\mathcal{S}_j$ can be found by interchanging in $\mathcal{S}_k$ the colors j and k $\quad (j \neq k, j \geq 2, k \geq 2)$.

Example 1: Two players, four colors.

$S_1 = \sum_{n=0}^{[\frac{N}{2}]}(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-3}{2}]} p_2p_3p_4(p_2^2 + p_3^2 + p_4^2)^n = 1 + (p_2^2 + p_3^2 + p_4^2)$; probability $p_1^2 + p_2^2 + p_3^2 + p_4^2$, inducing strategy: {2000,0200,0020,0002}: guess what you see.

$S_2 = \sum_{n=0}^{[\frac{N-1}{2}]} p_2(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-2}{2}]} p_3p_4(p_2^2 + p_3^2 + p_4^2)^n = p_2 + p_3p_4$; probability $2p_1p_2 + 2p_3p_4$; strategy: {1100, 0011}.

$\mathcal{S}_3 =$ {1010, 0101}; $\mathcal{S}_4 =$ {1001, 0110}

In the next examples we only look at S_1.

Example 2: Three players, four colors.

$S_1 = \sum_{n=0}^{[\frac{N}{2}]}(p_2^2 + p_3^2 + p_4^2)^n + \sum_{n=0}^{[\frac{N-3}{2}]} p_2p_3p_4(p_2^2 + p_3^2 + p_4^2)^n = 1 + (p_2^2 + p_3^2 + p_4^2) + p_2p_3p_4$; probability $p_1^3 + 3p_1(p_2^2 + p_3^2 + p_4^2) + 6p_2p_3p_4$; $\mathcal{S}_1$ ={3000, (1200, 1020, 1002), 0111}.

Example 3: Four players, four colors.
$S_1 = \sum_{n=0}^{[\frac{N}{2}]}(p_2^2+p_3^2+p_4^2)^n + \sum_{n=0}^{[\frac{N-3}{2}]} p_2p_3p_4(p_2^2+p_3^2+p_4^2)^n = 1 + (p_2^2+p_3^2+p_4^2) +$
$(p_2^2+p_3^2+p_4^2)^2 + p_2p_3p_4$; probability $p_1^4 + 6p_1^2(p_2^2+p_3^2+p_4^2) + (p_2^4+p_3^4+p_4^4) + 6(p_2^2p_3^2 +$
$+p_4^2p_2^2 + p_3^2p_4^2) + 24p_1p_2p_3p_4$.
$\mathcal{S}_1 =$ {4000, (2200,2020, 2002),(0400, 0040, 0004,0220,0202,0022), 1111}

Example 4: Five players, four colors.
$S_1 = \sum_{n=0}^{[\frac{N}{2}]}(p_2^2+p_3^2+p_4^2)^n + \sum_{n=0}^{[\frac{N-3}{2}]} p_2p_3p_4(p_2^2+p_3^2+p_4^2)^n = 1 + (p_2^2+p_3^2+p_4^2) +$
$(p_2^2+p_3^2+p_4^2)^2 + p_2p_3p_4 + p_2p_3p_4(p_2^2+p_3^2+p_4^2)$.
$\mathcal{S}_1 =${ 5000, (3200, 3020, 3002), (1400, 1040, 1004, 1220, 1202, 1022), 2111, (0311, 0131, 0113)}.

PART II

In this part N distinguishable players are fitted at random with a white or black hat, where the probabilities of getting a white or black hat (p respectively q; $p+q=1$) may be different, but known and the same to all the players.

II.1 Generation of BAD CASES by GOOD CASES.

In this part we give the white hat code 0 and the black hat code 1.
Each possible configuration of the white and black hats can be represented by an element of $B = \{b_1b_2 \dots b_N | b_i \in \{0,1\}, i = 1,2..,N\}$.
Player i sees binary code $b_1..b_{i-1}b_{i+1}..b_N$ with decimal value $s_i = \sum_{k=1}^{i-1} b_k.2^{N-k-1} + \sum_{k=i+1}^{N} b_k.2^{N-k}$.
Each player has to make a choice (independent of all other players) out of two possibilities: 0='guess white' and 1='guess black'. We define a decision matrix $D = (a_{i,j})$ where $i \in \{1,2,..,N\}$ (players); $j = s_i$; $a_{i,j} \epsilon \{0,1\}$ (guess white or guess black).
The meaning of $a_{i,j}$ is: player i sees j and takes decision $a_{i,j}$ (guess white or guess black).
We observe the total probability (sum) of our guesses.
For each $b_1b_2 \dots b_N$ in B with n zero's ($n \in \{0,1,..,N\}$) we have (start: sum=0):
CASE $b_1b_2 \dots b_N$
IF $a_{1,s_1} = b_1$ AND $a_{2,s_2} = b_2$ AND ... AND $a_{N,s_N} = b_N$ THEN sum=sum+p^nq^{N-n} ;
(all players guess right).
Any choice of the $a_{i,j}$ in the decision matrix determines which CASES have a positive contribution to sum (a GOOD CASE) and which CASES don't contribute positive to sum (a BAD CASE).
We focus on player i ($i \in \{1,..,N\}$). Each $a_{i,j} = 0$ has a counterpart $a_{i,j} = 1$ and vice versa: use the flipping procedure in position i: CASE $b_1 \dots b_i \dots b_N \rightarrow$ CASE $b_1 \dots 1-b_i \dots b_N$.
When we have a GOOD CASE with probability q^kp^{N-k} ($k \in \{0,1,..,N\}$) , then this single GOOD CASE generates (by a single bit flip) k BAD CASES, each with probability $q^{k-1}p^{N-k+1}$ and $N-k$ BAD CASES, each with probability $q^{k+1}p^{N-k-1}$. In short: 0^k1^{N-k} generates k of $0^{k-1}1^{N-k+1}$ and $N-k$ of $0^{k+1}1^{N-k-1}$.
We are interested in both 'left' generates $\mathcal{G}_L: \{0^{N-k}1^k\} \rightarrow \{0^{N-k+1}1^{k-1}\}$ and 'right' generates $\mathcal{G}_R: \{0^{N-k}1^k\} \rightarrow \{0^{N-k-1}1^{k+1}\}$, where we use the notation $\{0^m1^n\}$ for the set of all $\binom{m+n}{n}$ configurations of elements with m zero's and n one's.
For example: $\mathcal{G}_L$: 00011 →{00001, 00010}; $\mathcal{G}_R$: 00011 →{00111, 01011, 10011}.
When we have more than one GOOD CASE with probability q^kp^{N-k}, then the behavior of the number of generated BAD CASES is dependent of the order in which we choose the GOOD CASES.

We give an example: (see Table 3.1); N=5, k=2, $\mathcal{g}_L$; when we start with GOOD CASES={00011, 00101} we get generated BAD CASES: {00001,00010,00100}. But when we start with GOOD CASES={00011, 01100} then the generated BAD CASES are: {00001,00010,00100,01000}. In the next sections we show that optimal solutions of our game can be found by lexicographical ordering of the GOOD CASES.

i	$\{0^3 1^2\}$					$\rightarrow$	$\{0^4 1\}$										$e_{L,k,i}$	$E_{L,k,i}$	$\frac{E_{L,k,i}}{i}$
1	0	0	0	1	1	→	0	0	0	0	1	0	0	0	1	0	2	2	2.00
2	0	0	1	0	1	→	0	0	0	0	1	0	0	1	0	0	1	3	1.50
3	0	0	1	1	0	→	0	0	0	1	0	0	0	1	0	0	0	3	1.00
4	0	1	0	0	1	→	0	0	0	0	1	0	1	0	0	0	1	4	1.00
5	0	1	0	1	0	→	0	0	0	1	0	0	1	0	0	0	0	4	0.80
6	0	1	1	0	0	→	0	0	1	0	0	0	1	0	0	0	0	4	0.67
7	1	0	0	0	1	→	0	0	0	0	1	1	0	0	0	0	1	5	0.71
8	1	0	0	1	0	→	0	0	0	1	0	1	0	0	0	0	0	5	0.63
9	1	0	1	0	0	→	0	0	1	0	0	1	0	0	0	0	0	5	0.56
10	1	1	0	0	0	→	0	1	0	0	0	1	0	0	0	0	0	5	0.50

Table II.1.1: GOOD CASES are of the form $0^3 1^2$ (left block), the second and third block are the generated BAD CASES of the form $0^4 1$.

We define $e_{L,k,i}$ as the number of generated BAD CASES, applying $\mathcal{g}_L$, in row i of the $\binom{N}{k}$ possible configurations $0^k 1^{N-k}$. The cumulative number of generated BAD CASES is defined by $E_{L,k,i} = \sum_{j=1}^{i} e_{L,k,j}$. Analog we define $e_{R,k,i}$ and $E_{R,k,i}$.

The quotients $E_{L,k,i}/i$ and $E_{R,k,i}/i$ play a key role in our theory (see Theorem 5.1). We notice that in this example we have a minimal quotient $E_{L,k,i}/i$ at the last row.

II.2 Lexicographical ordering.

In this section we consider the hat problem with m white and n black hats. What we need is a lower bound of the total number of generated BAD CASES $E_{L,k,i}$, where we use $\mathcal{g}_L$ and i is the number of GOOD CASES with probability $q^k p^{m+n-k}$ $(1 \leq i \leq \binom{m+n}{n})$. We will show that lexicographical ordering is a way to obtain the lower bound. Elements of $\mathcal{g}_L(\{0^m 1^n\})$ at row i that occur for the first time contribute to $e_{L,k,i}$. Each element of $\mathcal{g}_L(\{0^m 1^n\})$ at row i that has been produced already is called a *hit* and doesn't contribute to $e_{L,k,i}$.

Lemma II.2.1

Hits in $\mathcal{g}_L\{0^m 1^n\}$ only occur when the Hamming distance in $\{0^m 1^n\}$ is 2.

Proof

The Hamming distance between $a \in \{0^m 1^n\}$ and $\mathcal{g}_L(a)$ is 1 (1 bit is flipped). When $\mathcal{g}_L(a) \cap \mathcal{g}_L(b) \neq \emptyset$ and $a \neq b$ then the Hamming distance between a and b is 2. Conversely, when the distance is 2, then bit flipping the 1's in the positions where the distance is generated will produce the hit.■

Lemma II.2.2

Minimizing the total number of generated BAD CASES is the same as maximizing the total number of hits.

Proof

What we need is a lower bound of the total number of generated BAD CASES $E_{L,k,i}$, where i is the number of GOOD CASES we have used. Let $H_{L,k,i}$ be the number of total hits in the first i rows.
For fixed i we have $E_{L,k,i} + H_{L,k,i} = ni$, so for each i we have: $min\ E_{L,k,i} = max\ H_{L,k,i}$. ■

Theorem II.2.1

Lexicographical ordering generates a lower bound on $E_{L,k,i}$ $(i = 1,2,..,\binom{m+n}{n})$, where we have m players with a white hat and n players with a black hat.

Proof

Combining Lemma's II.2.1 and II.2.2 yields: Minimizing the total number of generated BAD CASES in $\mathcal{G}_L(\{0^m1^n\})$ can be done by maximizing the total number of elements in $\{0^m1^n\}$ with Hamming distance 2. Without loss of generality we start with 0^m1^n. We focus on an ordered subset of i elements of $\{0^m1^n\}$, starting with 0^m1^n where $1 \leq i \leq \binom{m+n}{n}$. To obtain maximum number of elements with Hamming distance 2, we work in $0^{m-s}\{0^s1^n\}$, where $\{0^s1^n\}$ needs to be as small as possible. We have: $\binom{s+n-1}{n} < i \leq \binom{s+n}{n}$. When $1 \leq i \leq \binom{m+n-1}{n}$ we have to deal with $0\{0^{m-1}1^n\}$ and when $\binom{m+n-1}{n} < i \leq \binom{m+n}{n}$ we have $1\{0^m1^{n-1}\}$.

So we arrive at $[0\{0^{m-1}1^n\} \cup 1\{0^m1^{n-1}\}]$, where [ ] is the notation for an ordered set; all elements of $0\{0^{m-1}1^n\}$ precedes all elements of $1\{0^m1^{n-1}\}$.

We use induction to $m+n$. For two players $(m+n=2)$ we can verify that lexicographical ordering generates an optimal solution. Now suppose that for $m+n-1$ players lexicographical ordering generates an optimal solution. Then $\{0^{m-1}1^n\}$ and $\{0^m1^{n-1}\}$ on itself gives the desired result by induction assumption, but we have to study the interference in $[0\{0^{m-1}1^n\} \cup 1\{0^m1^{n-1}\}]$. We focus on the first i rows in $\{0^m1^n\}$. When $i \leq \binom{m+n-1}{n}$ then $0\{0^{m-1}1^n\}$ will do the job. Suppose $i > \binom{m+n-1}{n}$ then we use all of $0\{0^{m-1}1^n\}$ and $i - \binom{m+n-1}{n}$ elements of $1\{0^m1^{n-1}\}$.

We have $\mathcal{G}_L(1\{0^m1^{n-1}\}) = [0\{0^m1^{n-1}\} \cup 1\mathcal{G}_L(\{0^m1^{n-1}\})]$ where $0\{0^m1^{n-1}\} = 0\mathcal{G}_L(\{0^{m-1}1^n\}) = \mathcal{G}_L(0\{0^{m-1}1^n\})$ so *every* element of $0\{0^m1^{n-1}\}$ has a hit with $\mathcal{G}_L(0\{0^{m-1}1^n\})$ and this has no effect on $E_{L,k,i}$. The elements of $1\mathcal{G}_L(\{0^m1^{n-1}\})$ start with an 1 and have no interference with $0\{0^{m-1}1^n\}$. ■

In this section we focused on $\mathcal{G}_L$, but the same procedure can be applied to $\mathcal{G}_R$.

II.3 SAFE positions.

We start with a GOOD CASE with probability q^kp^{N-k} $(k \in \{0,1,2,..,N\})$. When $k = 0$ then p^Ngenerates via $\mathcal{G}_R$ all N elements qp^{N-1} ; when $k = N$ then q^N generates via $\mathcal{G}_L$ all N elements pq^{N-1}. In this section we suppose $k \in \{1,2,..,N-1\}$.

When $aq^{k-1}p^{N-k+1} + bq^kp^{N-k} + cq^{k+1}p^{N-k-1}$ is part of the solution of our hat problem, then we use the notation $[a,b,c]_k$. We want $[\binom{N}{k-1} - E_{L,k,i}, i, \binom{N}{k+1} - E_{R,k,i}]_k$ less than $[\binom{N}{k-1}, 0, \binom{N}{k+1}]_k$ when $i = 1,2,..,\binom{N}{k} - 1$. If this is true, we call it a SAFE position.

Theorem II.3.1

If $\frac{E_{L,k,i}}{i} > \frac{E_{L,k,\binom{N}{k}}}{\binom{N}{k}}$ and $\frac{E_{R,k,i}}{i} > \frac{E_{R,k,\binom{N}{k}}}{\binom{N}{k}}$ $(i = 1,2,..,\binom{N}{k} - 1)$, then we have a SAFE position.

Proof.

We have a SAFE position if $(\binom{N}{k-1} - E_{L,k,i})q^{k-1}p^{N-k+1} + iq^kp^{N-k} + (\binom{N}{k+1} - E_{R,k,i})q^{k+1}p^{N-k-1} < \binom{N}{k-1}q^{k-1}p^{N-k+1} + \binom{N}{k+1}q^{k+1}p^{N-k-1}$.

It follows: $-E_{L,k,i}q^{k-1}p^{N-k+1} + iq^kp^{N-k} - E_{R,k,i}q^{k+1}p^{N-k-1} < 0$

So: $-E_{L,k,i}\left(\frac{p}{q}\right)^2 + i\left(\frac{p}{q}\right) - E_{R,k,i} < 0$ when $i = 1,2,..,\binom{N}{k} - 1$.

What we need is $i^2 - 4E_{L,k,i}E_{R,k,i} < 0$. Using $\frac{E_{L,k,i}}{i} > \frac{E_{L,k,\binom{N}{k}}}{\binom{N}{k}} = \frac{\binom{N}{k-1}}{\binom{N}{k}} = \frac{k}{N-k+1}$ and $\frac{E_{R,k,i}}{i} > \frac{E_{R,k,\binom{N}{k}}}{\binom{N}{k}} = \frac{\binom{N}{k+1}}{\binom{N}{k}} = \frac{N-k}{k+1}$ we get: $i^2 - 4E_{L,k,i}E_{R,k,i} < \frac{E_{L,k,i}E_{R,k,i}}{k(N-k)}\{(k+1)(N-k+1) - 4k(N-k)\} = \frac{E_{L,k,i}E_{R,k,i}}{k(N-k)}\{(1-3k)(N-k) + k + 1)\} \leq \frac{E_{L,k,i}E_{R,k,i}}{k(N-k)}\{(1-3k) + k + 1)\} = \frac{E_{L,k,i}E_{R,k,i}}{k(N-k)}\{2-2k\} \leq 0$ $(k = 1,2,..,N-1)$. ■

Until now we studied the behaviour of an isolated set $aq^{k-1}p^{N-k+1} + bq^kp^{N-k} + cq^{k+1}p^{N-k-1}$. In practice there is a much more complicated interference, but the number of generated items can only grow, so that $i^2 - 4E_{L,k,i}E_{R,k,i} < 0$ will hold in general when we can prove it for the isolated case.
In the next section we will prove that lexicographical order will produce SAFE positions.

II.4 Lexicographical order and SAFE positions.

Theorem II.4.1
Lexicographical order produces SAFE positions.
Proof
We start with $\mathcal{G}_L$.
We use induction to the number of players N.
N=2 is straightforward.
Suppose we have proven the theorem for $N-1$ players.
We have $\{0^{N-k}1^k\} = 0\{0^{N-k-1}1^k\} \cup 1\{0^{N-k}1^{k-1}\}$ and therefore $lex\{0^{N-k}1^k\} = [lex0\{0^{N-k-1}1^k\} \cup lex1\{0^{N-k}1^{k-1}\}]_k$, where lex stands for lexicographical order.
$0\{0^{N-k-1}1^k\}$ consists of $n_1 = \binom{N-1}{k}$ elements and $\mathcal{G}_L(0\{0^{N-k-1}1^k\}) = 0\mathcal{G}_L(\{0^{N-k-1}1^k\})$, where $\mathcal{G}_L$ is now working on $N-1$ players and by induction assumption we have: $\frac{A_i}{i} > \frac{A_{n_1}}{n_1}$ where A_i stands for total number of generated BAD CASES up to row i and $i = 1,2,..,n_1 - 1$.
$\mathcal{G}_L(\{0^{N-k-1}1^k\}) = \{0^{N-k}1^{k-1}\}$, so $A_{n_1} = \binom{N-1}{k-1}$.
$1\{0^{N-k}1^{k-1}\}$ consists of $n_2 = \binom{N-1}{k-1}$ elements and $\mathcal{G}_L(1\{0^{N-k}1^{k-1}\}) = 0\{0^{N-k}1^{k-1}\} + 1\mathcal{G}_L(\{0^{N-k}1^{k-1}\})$, where $0\{0^{N-k}1^{k-1}\}$ has already been produced by $\mathcal{G}_L(0\{0^{N-k-1}1^k\})$ and therefore has no contribution to the number of generated BAD CASES. We can concentrate on $1\mathcal{G}_L(\{0^{N-k}1^{k-1}\})$ with no overlap with $0\mathcal{G}_L(\{0^{N-k-1}1^k\})$; $1\mathcal{G}_L(\{0^{N-k}1^{k-1}\})$ is working on $N-1$ players and by induction we have: $\frac{B_i}{i} > \frac{B_{n_2}}{n_2}$ where B_i stands for total number of generated BAD CASES up to row i (in sector $1\{0^{N-k}1^{k-1}\}$) and $i = 1,2,..,n_2 - 1$.
$\mathcal{G}_L(\{0^{N-k}1^{k-1}\}) = \{0^{N-k+1}1^{k-2}\}$, so $B_{n_2} = \binom{N-1}{k-2}$.
$\frac{B_i}{i} > \frac{B_{n_2}}{n_2}$ is only valid in the set $1\{0^{N-k}1^{k-1}\}$. We have $B_i > i\frac{B_{n_2}}{n_2} = i\left(\frac{k-1}{N-k+1}\right)$
We have to prove:
$\frac{C_{n_1+i}}{n_1+i} > \frac{C_{n_3}}{n_3}$ $(i = 1,2,..,n_2 - 1)$ where C_{n_1+i} stands for the generated BAD CASES by $\mathcal{G}_L(0\{0^{N-k-1}1^k\})$ plus total number of generated BAD CASES up to row i in set $1\{0^{N-k}1^{k-1}\}$,
$C_{n_1+i} = A_{n_1} + B_i$.
where $i = 1,2,..,n_2 - 1$; $n_3 = n_1 + n_2 = \binom{N}{k}$; $C_{n_3} = C_{n_1} + C_{n_2} = \binom{N}{k-1}$.
$\frac{C_{n_1+i}}{n_1+i} = \frac{A_{n_1}+B_i}{n_1+i} > \frac{C_{n_3}}{n_3} = \frac{k}{N-k+1}$; we have:

$(N-k+1)\left(A_{n_1}+B_i\right)-k(n_1+i) > (N-k+1)\binom{N-1}{k-1}+(k-1)i-k\left(\binom{N-1}{k}+i\right) =$
$\binom{N-1}{k-1}\{(N-k+1)-(N-k)\}-i > 0$ when $i < \binom{N-1}{k-1} = n_2$.

We focus on $\mathcal{G}_R$. By symmetry we have: $\mathcal{G}_R(\{0^m1^n\}) = \mathcal{G}_L(\{0^n1^m\})$, so we get SAFE positions.■

II.5 Optimal strategies and maximal winning probabilities.

Lemma II.5.1

The sequence $\binom{N}{k}p^{N-k}q^k$ $(k=0,1,..,N)$ consists of two parts: the first part is monotone increasing (may be empty) and the second part is monotone decreasing (may be empty).

Proof

Increasing values for $k=1,2,..,s$ will be found if: $\frac{\binom{N}{k}p^{N-k}q^k}{\binom{N}{k-1}p^{N-k+1}q^{k-1}} = \frac{(N-k+1)}{k}\frac{q}{p} > 1$, so $p < \frac{N-(k-1)}{N+1}$. It follows: $p < \frac{N-(s-1)}{N+1}$. In the same way we can show that decreasing values for $k = s+1, s+2,..,N$ comes to $p > \frac{N-(k-1)}{N+1}$. So: $p > \frac{N-s}{N+1}$.

$\forall_{p\in(0,1)}\exists_{s\in\{0,1,..,N\}}$: $\frac{N-s}{N+1} < p < \frac{N-(s-1)}{N+1}$ and for this value of s we have:
sequence is increasing if $k=1,2,..,s$ and decreasing if $k=s+1,s+2,..,N$. ■

It follows from sections II.3 and II.4 that optimal solutions can be found by focusing on $[\binom{N}{k-1},0,\binom{N}{k+1}]_k$.

We are looking for an optimal spread of these $[\binom{N}{k-1},0,\binom{N}{k+1}]_k$ configurations. We define
$\mathcal{W}_k = [\binom{N}{k-1},0,\binom{N}{k+1}]_k = \binom{N}{k-1}q^{k-1}p^{N-k+1} + 0.q^kp^{N-k} + \binom{N}{k+1}q^{k+1}p^{N-k-1}$
$(k=1,2,..,N-1)$.

We define 0-parity of a CASE as the parity of the number of zero's in that CASE.

Theorem II.5.1

The maximal probabilities and the optimal strategies of our hat problem are:
when $p < q$ or N is even:

$\sum_{k\ even}\binom{N}{k}p^kq^{N-k} = \frac{1+(q-p)^N}{2}$; strategy: even 0-parity,

when $p > q$ and N is odd:

$\sum_{k\ odd}\binom{N}{k}p^kq^{N-k} = \frac{1-(q-p)^N}{2}$; strategy: odd 0-parity,

when $p = q$:

$\frac{1}{2}$; strategy: all players odd 0-parity or all players even 0-parity.

Proof

In an optimal solution there can't be much distance between the $\mathcal{W}_k$. For instance, if $\mathcal{W}_k$ and $\mathcal{W}_{k+4}$ are in an optimal solution, then we can insert without interference $\mathcal{W}_{k+2}$. Furthermore if $\mathcal{W}_k$ and $\mathcal{W}_{k+3}$ are in an optimal solution and there is nothing between we can improve the optimal solution by replacing $\mathcal{W}_k$ by $\mathcal{W}_{k+1}$ or $\mathcal{W}_{k+3}$ by $\mathcal{W}_{k+2}$ (use Lemma II.5.1). So in the optimal solution we have $\mathcal{W}_k$, $\mathcal{W}_{k+2}$, ...: all even or all odd terms in $\sum_k\binom{N}{k}p^kq^{N-k}$.

We have: $\sum_{k\ even}\binom{N}{k}p^kq^{N-k} - \sum_{k\ odd}\binom{N}{k}p^kq^{N-k} = (q-p)^N$.

Using $\sum_{k\ even}\binom{N}{k}p^kq^{N-k} + \sum_{k\ odd}\binom{N}{k}p^kq^{N-k} = (q+p)^N = 1$, we obtain our goal. ■

So we have: *maximal winning probability:* $\frac{1+|q-p|^N}{2}$ $(N=1,2,3,\ldots)$.

II.6 Hats: Nothing.

Until now we were interested in perfect guessing: all players must guess correct. In this section we demand the opposite: *all players must guess wrong*.
The theory of maximal winning probabilities stays the same, but the optimal strategy is now: use the strategy in 'all players guess right', followed by a bit flip.

Appendix A (part II)

In this appendix we prove some properties of the generated BAD CASES, using lexicographical order in $\{0^m1^n\}$. We define $L(m,n) = \mathcal{G}_L(lex(\{0^m1^n\})$ and $R(m,n) = \mathcal{G}_R(lex(\{0^m1^n\})$.
L is the ordered set of all generated items e_i: $L(m,n) = [e_1, e_2, \dots, e_{\binom{m+n}{n}}]$ where we use the notation [] for an ordered set. For example: $L(3,2) = [2,1,0,1,0,0,1,0,0,0]$ (see Table II.1.1).

Lemma A.1
$lex(\{0^m1^n\}) = [\bigcup_{k=0}^{n} 1^k 0 lex(\{0^{m-1}1^{n-k}\})]$
Proof
Every element of $\{0^m1^n\}$ starts with a fixed number of one's followed by a zero. ∎

Lemma A.2
$\mathcal{G}_L(1^k)0\{0^{m-1}1^{n-k}\} \subset \mathcal{G}_L\left([\bigcup_{s=0}^{k-1} 1^s 0\{0^{m-1}1^{n-s}\}]\right)$
Proof
$\mathcal{G}_L(1^k)0\{0^{m-1}1^{n-k}\} = [\bigcup_{s=0}^{k-1} 1^s 0 1^{k-s-1} 0\{0^{m-1}1^{n-k}\}] \subset \mathcal{G}_L\left([\bigcup_{s=0}^{k-1} 1^s 0\{0^{m-1}1^{n-s}\}]\right)$. ∎

Theorem A.1
$L(m,n) = [\bigcup_{k=0}^{n} L(m-1, n-k)]$
Proof
Using Lemma A.1 we have: $L(m,n) = \mathcal{G}_L(lex(\{0^m1^n\}) = \mathcal{G}_L\left([\bigcup_{k=0}^{n} 1^k 0 lex\{0^{m-1}1^{n-k}\}]\right) =$
$[\bigcup_{k=0}^{n} \mathcal{G}_L(1^k) 0 lex(\{0^{m-1}1^{n-k}\}) \bigcup_{k=0}^{n} 1^k 0 \mathcal{G}_L(lex(\{0^{m-1}1^{n-k}\}))]$.
By Lemma A.2 we have $\mathcal{G}_L(1^k) 0 lex(\{0^{m-1}1^{n-k}\}$ is hit by preceding lexicographical elements, so we have: $L(m,n) = [\bigcup_{k=0}^{n} 1^k 0 \mathcal{G}_L\left(lex(\{0^{m-1}1^{n-k}\})\right] = [\bigcup_{k=0}^{n} L(m-1, n-k)]$. ∎

Theorem A.2
$$L(m,n) = [n \bigcup_{k_{n-1}=0}^{m-1} (n-1) \bigcup_{k_{n-2}=0}^{k_{n-1}} (n-2) \dots \dots 3 \bigcup_{k_2=0}^{k_3} 2 \bigcup_{k_1=0}^{k_2} \left(10^{k_1+1}\right)] \qquad (*)$$
Proof
Induction to $m+n$.
We have $L(0,n) = [n]$ and $L(m,1) = [10^m]$, which both agrees with $(*)$.
We also have:
$[n \bigcup_{k_{n-1}=0}^{m-1} (n-1) \bigcup_{k_{n-2}=0}^{k_{n-1}} (n-2) \dots \dots 3 \bigcup_{k_2=0}^{k_3} 2 \bigcup_{k_1=0}^{k_2} \left(10^{k_1+1}\right)] =$
$[n \bigcup_{k_{n-1}=0}^{m-2} (n-1) \bigcup_{k_{n-2}=0}^{k_{n-1}} (n-2) \dots \dots 3 \bigcup_{k_2=0}^{k_3} 2 \bigcup_{k_1=0}^{k_2} \left(10^{k_1+1}\right) +$
$(n-1) \bigcup_{k_{n-2}=0}^{m-1} (n-2) \dots \dots 3 \bigcup_{k_2=0}^{k_3} 2 \bigcup_{k_1=0}^{k_2} \left(10^{k_1+1}\right)] =$ (induction assumption)=
$[L(m-1,n) \cup L(m,n-1)]$ =(Theorem A.1)= $[L(m-1,n) \bigcup_{k=0}^{n-1} L(m-1, n-1-k)] =$
(Theorem A.1)= $L(m,n)$ ∎

Theorem A.3
$R(n,m) = L(m,n)$
Proof
By reflection in the middle of our set of players and simultaneously interchanging the color of our players we obtain: $R(n,m) = L(m,n)$. ∎

References:

1. W. Blum, *Denksport für Hutträger*, Die Zeit, May 3, 2001.
2. J. Buhler, *Hat tricks*, Math. Intelligencer 24 (2002), 44-49.
3. E. Burke, S. Gustafson, G. Kendall, *A puzzle to challenge genetic programming*, Genetic Programming, 136-147, Lecture Notes in Computer Science, Springer, 2002.
4. S. Butler, M. Hajianghayi, R Kleinberg, T. Leighton, *Hat guessing games*, SIAM J Discrete Math 22 (2008), 592-605.
5. G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, *Covering Codes*, North-Holland, Mathematical Library 54, 1997.
6. T. Ebert, *Applications of recursive operators to randomness and complexity*, Ph.D. Thesis, University of California, Santa Barbara, 1998.
7. T. Ebert, W. Merkle, H. Vollmer, *On the autoreducibility of random sequences*, SIAM J. Comp. 32 (2003), 1542-1569.
8. M. Gardner, *The 2nd Scientific American Book of Mathematical Puzzles&Diversions*, Simon and Schuster, New York, 1961.
9. W. Guo, S. Kasala, M. Rao, B. Tucker, *The hat problem and some variations*, Advances in distribution theory, order statistics, and inference, 459-479, Stat. Ind. Technol., Birkhäuser Boston, 2006.
10. C. Hardin, A. Taylor, *The Mathematics of coordinated inference: A study of generalized hat problems*, Springer International Publishing Switzerland 2013.
11. B. Johnson, http://mathstat.slu.edu/~johnson/public/maths/hatproblem.pdf
12. G. Kéri, *Tables for bounds on covering codes*, www.sztaki.hu/~keri/codes
13. M. Krzywkowski, *On the hat problem, its variations, and their applications*, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica 9 (1), 55-67, 2010.
14. M Krzywkowski, *A more colorful hat problem*, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica 10 (1), 67-77, 2011.
15. H. Lenstra, G. Seroussi, *On hats and other covers*, (Extended Summary), arXiv:cs/0509045v1 [cs.IT] 15 sep 2005.
16. J. Poulos, *Could you solve this $1 million hat trick?*, abcNews, November 29, 2001
17. S. Robinson, *Why mathematicians now care about their hat color*, The New York Times, Science Times Edition, page D5, April 10, 2001.
18. T. van Uem, *The Hat Game and covering codes*, https://arxiv.org/abs/1612.00276 1 dec 2016
19. T. van Uem, *Asymmetric Hat Game with three players and three colors*, https://arxiv.org/abs/1612.05924v2 6 jan 2017
20. T. van Uem, *General three and four person two color Hat Game*, https://arxiv.org/abs/1704.04244 13 apr 2017
21. T. van Uem, *Hats: all or nothing*, https://arxiv.org/abs/1801.01512v2 26 apr 2018
22. P. Winkler, *Mathematical Mind-Benders*, A.K. Peters, Wellesley, Massachusetts, 2007